Larmor precession reexamined: Testable correction and its ramifications

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A rigorous treatment is given for solving the full Schrodinger equation for a spin-polarized plane wave passing through a region of uniform magnetic field, thereby providing a modified formula for Larmor precession, which, in turn, reduces to the standard expression in certain regimes of the experimental parameters. We show that there are experimentally verifiable regimes of departure from the standard formula. The treatment is then extended to the case of a spin-polarized wave packet passing through a uniform magnetic field. The results obtained from the standard expression for Larmor precession and that obtained from the modified formula are compared in various regimes of the experimental parameters. Implications of such an exact treatment of Larmor precession are indicated in the contexts of modeling a quantum clock as well as in the testing of single particle Bell's inequality.

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I. INTRODUCTION

If a spin-$1/2$ particle passes through a region of uniform magnetic field, it is well known that the time evolution of the spin of the particle undergoes what is commonly known as Larmor precession. For example, if the particle with an initial spin orientation along the $+\hat{x}$ axis passes through a magnetic field oriented along the $+\hat{z}$ axis, the spin precesses in the x-y plane with a frequency determined by the strength of the magnetic field and the magnetic moment of the particle. This frequency is known as the Larmor frequency - a topic which is not only commonly discussed in any undergraduate quantum mechanics course, but also has wide applications, particularly in the analysis of experiments involving neutron, electron and atomic interferometry. One of the important applications of Larmor precession is its use in calculating the tunneling time through a potential barrier where the time during which the particle passes through the barrier can be predicted via the Larmor precession relation that provides a correspondence between the transit time and the angle of rotation of the spin orientation of the spin-$1/2$ particles [1–5].

However, surprisingly, the treatments given in all the standard text-books[6–12] on the subject are fundamentally inadequate because none of them considers the solution of the full Schrodinger equation in this problem involving both the spin and the space parts. In particular, the usual treatments ignore the spatial part altogether and write the Hamiltonian only in terms of the potential energy arising out of the spin-magnetic field interaction. For example, Cohen-Tannoudji et al.[9] provide their treatment of Larmor precession by considering only the spin dynamics of the system in the presence of the uniform magnetic field. On the other hand, Bohm[10] gives a semi-classical argument as the basis for Larmor precession. Another line of justification for this procedure assumes that the particle is at rest while it interacts with the magnetic field, an obviously untenable premise that violates the uncertainty principle as the spatial extent of the magnetic field is not infinite[11].

A specious argument for ignoring the kinetic energy term in the Hamiltonian would be to take this term to be much smaller in magnitude as compared to the spin-magnetic field potential energy term. To check whether this assumption is valid or not, it is important to note that most experiments involving Larmor precession are done at or near room temperature. At this ambient temperature, the average kinetic energy of the particles under study is of the order of kT. At room temperature the value of kT is about $0.025eV$. Let us now look at a typical value of the spin-magnetic field interaction energy. A magnetic field of $1T$ is achievable in laboratory conditions and is the typical value of the magnetic field in the relevant experiments. For this strength of the magnetic field, the interaction energy arising due to the interaction of the field with the spin magnetic moment of a neutron is of the order of $5 \times 10^{-8} eV$. Comparing this with the value of the kinetic energy, obviously, the assumption that the kinetic energy of the particle is negligible compared to the spin-magnetic field interaction energy in typical experiments does not hold good. In fact, under such conditions, the kinetic energy term is much larger than the potential energy term.

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Here we note that all the treatments of calculating tunneling time through a barrier based on Larmor clock [1–5] pertain to a spatial region within which a constant magnetic field and an external potential V_0 are both confined. It is essentially due to the presence of the external potential that both spin-up and spin-down particles see potential barriers, but of different heights. Interestingly, if one takes the limit of the external potential $V_0 \to 0$, the usual Larmor precession relation cannot be recovered even in the presence of high magnetic field. On the other hand, in our treatment, we do not consider any external potential so that while a spin-up particle sees a potential well, a spin-down particle sees a barrier. Hence, our treatment is completely different from all the prevalent treatments pertaining to tunneling time calculation using Larmor clock.

We begin by considering a wave function whose space part is a plane wave and is spin polarized in the $+x$ direction. By examining closely the time evolution of the entire wave function, that is, both the spin and the space parts, caused by the interaction of the spin of the particle with the magnetic field, we find an interesting feature - due to the specifics of this spin-magnetic field interaction, it is possible to derive the time evolution of the entire wave function by solving the Schrodinger equation for only the spatial part. Then, from the time-evolved entire wave function, one can find the change in the spin part of the wave function; this is shown in Section II. Subsequently, in Section III, the limit in which the result of our treatment matches the result of the standard Larmor precession as well as the limit in which there is an appreciable departure from Larmor precession are discussed. This treatment reveals that it is, in fact, the limit where the kinetic energy is much higher than the potential energy due to the spin-field interaction that the standard expression for Larmor precession holds true. In Section III numerical estimates of departure from the standard Larmor precession are presented. In Section IV we generalize the treatment given above to the case of an incident wave packet rather than a plane wave.

II. PARTICLE IN A UNIFORM MAGNETIC FIELD

Let us consider that the total initial wave function of a particle is represented by $\Psi_i = \psi_0 \otimes \chi$, where $\psi_0 = Ae^{ikx}$ is the spatial part taken to be a plane wave with wave number k, and $\chi = (\frac{1}{\sqrt{2}}(|\uparrow\rangle_z + |\downarrow\rangle_z)$ is the spin state polarized in the $+x$ direction with $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ are the eigenstates of $\widehat{\sigma}_z$. Now, we consider that the particle passes through a bounded region containing constant magnetic field directed along the $+z$-axis.

The interaction Hamiltonian is $H_{int} = \mu\vec{\sigma}.\mathbf{B}$ where μ is the magnetic moment of the neutron, $\mathbf{B}$ is the inhomogeneous magnetic field and $\vec{\sigma}$ is the Pauli spin matrices vector. Then the time evolved total wave function at $t = \tau$ after the interaction of spins with the uniform magnetic field is given by

$$\begin{aligned}\Psi(\mathbf{x},\tau) &= \exp(-\frac{iH\tau}{\hbar})\Psi(\mathbf{x},0)\\ &= \frac{1}{\sqrt{2}}\left[\psi^+(\mathbf{x},\tau)\otimes|\uparrow\rangle_z + \psi^-(\mathbf{x},\tau)\otimes|\downarrow\rangle_z\right] \quad (1)\end{aligned}$$

where $\psi^+(\mathbf{x},\tau)$ and $\psi^-(\mathbf{x},\tau)$ are the two components of the spinor $\psi = \begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}$ which satisfies the Pauli equation. The homogeneous magnetic field is written as $\mathbf{B} = B\widehat{z}$. The two-component Pauli equation can then be written as two coupled equations for ψ^+ and ψ^-, given by

$$i\hbar\frac{\partial\psi^+}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^+ - \mu B\widehat{z}\psi^+ \quad (2)$$

$$i\hbar\frac{\partial\psi^-}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^- + \mu B\widehat{z}\psi^- \quad (3)$$

Eqs.(2) and (3) imply that while a neutron having spin-up interacts with the spin rotator containing constant magnetic field, the associated spatial wave function (ψ^+) evolves under a *potential well* that has been generated due to the spin-magnetic field interaction; the associated spatial wave function (ψ^-) of a spin-down neutron evolves under a *potential barrier*. Before proceeding to focus on the solutions of Eqs. (2) and (3), we revisit in the next section the usual textbook treatment of this problem purporting to derive the Larmor precession relation for the rotated spin state after the spin-magnetic field interaction within the spin-rotator.

A. The usual textbook treatment of Larmor precession

The behaviour of the wave function after it starts interacting with the spin rotator is usually described by taking into account *only* the spin part of the wave function, while the space part is completely left out of the analysis. Neglecting the kinetic energy, the Hamiltonian of the system inside the spin rotator is taken to be $H = \mu\vec{\sigma}.\mathbf{B}$. The spin up and spin down parts of the wave function evolve according to the Schroedinger equation in the following way

$$i\hbar\frac{\partial\psi^+}{\partial t} = -\mu B\psi^+ \quad (4)$$

$$i\hbar\frac{\partial\psi^-}{\partial t} = +\mu B\psi^- \quad (5)$$

The solutions of the above two first order differential equations are $\psi^{\pm} = \psi_0 e^{\mp i\omega\tau}$ where $\omega = \mu B/\hbar$, and τ is the time over which the spin-magnetic field interaction

takes place. Putting these solutions in Eq.(1), we finally obtain

$$\Psi(x,\tau) = \psi_0 \frac{1}{\sqrt{2}} \left(e^{-i\omega\tau} \left|\uparrow\right\rangle + e^{i\omega\tau} \left|\downarrow\right\rangle \right) \tag{6}$$

Eq.(6) can be written as

$$\Psi(x,\phi) = \psi_0 \frac{e^{-i\phi/2}}{\sqrt{2}} \left(\left|\uparrow\right\rangle_z + e^{i\phi} \left|\downarrow\right\rangle_z \right) \equiv e^{-i\phi/2} \psi_0 \chi(\phi) \tag{7}$$

where

$$\phi = 2\omega\tau \tag{8}$$

is the well-known Larmor precession relation and ω is the Larmor frequency and, $e^{-i\phi/2}$ is the global phase. The spin state after the interaction is given by $\chi(\phi) = \frac{1}{\sqrt{2}} \left(\left|\uparrow\right\rangle + e^{i\phi} \left|\downarrow\right\rangle \right)$.

In the above treatment τ is taken to be the transit time through the spin rotator, given by $\tau = \frac{a}{v}$ where a is the spatial extension of spin-rotator containing the uniform magnetic field, and $v = \frac{\hbar k}{m}$ is the initial velocity of particle.

However, the above treatment has an intrinsic incompleteness because it does *not* take into account the evolution of spatial part, and hence the kinetic energy associated with the wave function. The detailed analysis of this problem should include the evolutions of both the spin and space parts of the total initial wave function. In the following section we fill this gap by giving a rigorous analysis of this problem.

B. The complete analysis of particle in a constant magnetic field problem

Here we begin by recalling that Eqs. (2) and (3) imply that in this problem, we effectively have a situation where the spin up part of the wave function faces a *potential well* while the spin down part of the wave function faces a *potential barrier*. This, in turn, entails that the information about the spin part of the wave function enters the space part of the wave function through this potential, and thus solving the Schroedinger equation for only the space part of the wave function suffices to get complete information about the time evolution of the combined spin-space wave function. Here we may stress once again that this method of treatment of the quantum mechanical problem of spin precession in a uniform magnetic field has remain unexplored. Therefore, instead of equations (4) and (5), one needs to solve Eqs.(2) and (3) explicitly.

The solutions to these equations, given our initial state, consist of a reflected part traveling in the $-x$-direction and a transmitted part of the wave function, traveling in the $+x$-direction. We should note here that the reflected part of the wave function exists *only* to the left of the spin rotator, and the transmitted part exists *only* to the right of the spin rotator. Our ultimate objective is to calculate the observable rotation of the spin part of the wave function caused by evolution of the state due to the spin-magnetic field interaction within the spin rotator.

For this, we need to look only at that part of the wave function which pertains to those neutrons which have actually passed through the spin rotator, or in other words we focus only on the transmitted part of the wave function. Therefore, using the solutions of equations (2) and (3), we will finally end up with the following state

$$\left|\Psi_f\right\rangle = \frac{N}{\sqrt{2}} (\psi_T^+ \left|\uparrow\right\rangle + \psi_T^- \left|\downarrow\right\rangle) \tag{9}$$

where N is the normalized constant can be written as $N = \int_v (|\psi_T^+|^2 + |\psi_T^-|^2) dv$.

Note that, Eq.(9) is an entangled state between the spin and spatial degrees of freedom of the transmitted part. The expressions for the reflected and transmitted parts of a wave function at a potential well or barrier are well known. However, from an empirical perspective, we note that if in the above setup, even if we use low energy or ultra-cold neutrons, having kinetic energy of the order of $5 \times 10^{-7} eV$, for the potential energy term $(|\mu B|)$ to exceed the kinetic energy term (E), we will need a field of the order of $10T$. Magnetic fields of such high intensity are difficult to produce in laboratory conditions, and therefore for all practical purposes, we should consider the situation where $E > \mu B$.

Now, since ψ^- evolves under a potential barrier confined between $x = 0$ and $x = a$ the reflected and tranmitted ports are respectively given by

$$\psi_R^- = A e^{-ikx} \frac{(k^2 - k_1^2)(1 - e^{2ik_1 a})}{(k+k_1)^2 - (k-k_1)^2 e^{2ik_1 a}} \tag{10}$$

$$\psi_T^- = A e^{ikx} \frac{4kk_1 e^{-ika} e^{ik_1 a}}{(k+k_1)^2 - (k-k_1)^2 e^{2ik_1 a}} \tag{11}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$, $k_1 = \frac{\sqrt{2m(E-\mu B)}}{\hbar}$ and a is the width of the spin rotator arrangement which contains the uniform magnetic field. The wave function ψ^+ evolves under a potential well, the expressions for the transmitted and the reflected part by replacing all the k_1's in Eqs.(10) and (11) by k_2 where $k_2 = \frac{\sqrt{2m(E+\mu B)}}{\hbar}$.

$$\psi_R^+ = A e^{-ikx} \frac{(k^2 - k_2^2)(1 - e^{2ik_2 a})}{(k+k_2)^2 - (k-k_2)^2 e^{2ik_2 a}} \tag{12}$$

$$\psi_T^+ = A e^{ikx} \frac{4kk_2 e^{-ika} e^{ik_2 a}}{(k+k_2)^2 - (k-k_2)^2 e^{2ik_2 a}} \tag{13}$$

Then, in the regime $E > \mu B$, we now rewrite equation (7) in the following form, which is the modified formula for

Larmor precession calculated by the explicit time-evolved solution of the spatial parts, ψ^+ and ψ^-

$$|\Psi_f\rangle = \frac{Ae^{ikx}}{\sqrt{2}}(ae^{i\phi_1}|\uparrow\rangle + be^{i\phi_2}|\downarrow\rangle) \equiv \psi_0\chi(\phi) \quad (14)$$

Here

$$a = \sqrt{Re(\psi_T^+)^2 + Im(\psi_T^+)^2} \quad (15)$$

$$b = \sqrt{Re(\psi_T^-)^2 + Im(\psi_T^-)^2} \quad (16)$$

$$\phi_1 = tan^{-1}\frac{Im(\psi_T^+)}{Re(\psi_T^+)} \quad (17)$$

$$\phi_2 = tan^{-1}\frac{Im(\psi_T^-)}{Re(\psi_T^-)} \quad (18)$$

where, by using equations (11) and (13), we find that

$$Re(\psi_T^-) = \frac{8kk_1(k^2+k_1^2)sin(ka)sin(k_1a) + 16k^2k_1^2cos(ka)cos(k_1a)}{(k+k_1)^4 + (k-k_1)^4 - 2(k+k_1)^2(k-k_1)^2cos(2k_1a)} \quad (19)$$

$$Im(\psi_T^-) = \frac{8kk_1(k^2+k_1^2)cos(ka)sin(k_1a) - 16k^2k_1^2sin(ka)cos(k_1a)}{(k+k_1)^4 + (k-k_1)^4 - 2(k+k_1)^2(k-k_1)^2cos(2k_1a)} \quad (20)$$

$$Re(\psi_T^+) = \frac{8kk_2(k^2+k_2^2)sin(ka)sin(k_2a) + 16k^2k_2^2cos(ka)cos(k_2a)}{(k+k_2)^4 + (k-k_2)^4 - 2(k+k_2)^2(k-k_2)^2cos(2k_2a)} \quad (21)$$

$$Im(\psi_T^+) = \frac{8kk_2(k^2+k_2^2)cos(ka)sin(k_2a) - 16k^2k_2^2sin(ka)cos(k_2a)}{(k+k_2)^4 + (k-k_2)^4 - 2(k+k_2)^2(k-k_2)^2cos(2k_2a)} \quad (22)$$

III. LIMITS OF VALIDITY OF THE STANDARD FORMULA FOR LARMOR PRECESSION

Let us now examine in what limit the above expressions do indeed reduce to the standard expressions for Larmor precession. As already mentioned, we are working in the range $E > \mu B$. Let us now consider the stronger limit where $E >> \mu B$. In this limit, the kinetic energy term of the Hamiltonian is appreciably larger than the potential energy term, and then, effectively, the time evolution of the entire wave function occurs due to a very shallow well and a very low barrier. This situation would correspond to the entire wave being transmitted, but picking up a phase. From the expressions for k, k_1, k_2, in the limit $E >> \mu B$, we find that $k \approx k_1 \approx k_2$.

In order to get the standard expression for Larmor precession, we will first set $k = k_1 = k_2$ in Eqs.(19),(20), (21), and (22) except when they appear inside sine or cosine functions, since the latter terms are much more sensitive to the differences in values of k, k_1, k_2. Eqs. (19) and (20) then simplify to

$$Re(\psi_T^-) = sin(ka)sin(k_1a) + cos(ka)cos(k_1a) \quad (23)$$

$$Im(\psi_T^-) = sin(k_1a)cos(ka) - cos(k_1a)sin(ka) \quad (24)$$

Using the above expressions in Eqs. (15) and (16), we find that $D = 1$ and $\phi_2 = (k_1 - k)a$. Similarly, rewriting equations (21) and (22), and using it in Eq.(14) and (16), we get $C = 1$ and $\phi_1 = (k_2 - k)a$. Therefore Eq.(14) now has the form

$$|\Psi_f\rangle = \frac{Ae^{ikx}}{\sqrt{2}}(e^{i(k_2-k)a}|\uparrow\rangle + e^{i(k_1-k)a}|\downarrow\rangle) \quad (25)$$

Since we have already stipulated the condition $k \approx k_1 \approx k_2$, we can binomially expand k_1 and k_2 around k and keep terms to the order of $\frac{\mu B}{E}$. Then $(k_1 - k)a = -k\frac{\mu B}{2E}a$. Using the relations $k = \frac{\sqrt{2mE}}{\hbar}$ and $v = \frac{\hbar k}{m}$, we can write $(k_1 - k)a = -\frac{\mu B}{\hbar}\frac{a}{v} = \omega t$. Similarly, $(k_2 - k)a = \frac{\mu B}{\hbar}\frac{a}{v} = -\omega t$. Therefore, we can write Eq.(25) as

$$\begin{aligned} |\Psi_f\rangle &= \frac{Ae^{ikx}}{\sqrt{2}}(e^{-i\omega t}|\uparrow\rangle + e^{i\omega t}|\downarrow\rangle) \quad (26) \\ &= \psi_0\frac{e^{-i\phi/2}}{\sqrt{2}}\left(|\uparrow\rangle_z + e^{i\phi}|\downarrow\rangle_z\right) \equiv e^{-i\phi/2}\psi_0\chi(\phi) \end{aligned}$$

which is exactly the equation we get from the spin-only treatment of the problem.

The above treatment brings out the curious feature that while the standard treatment ignores the kinetic energy term of the Hamiltonian, on solving the problem in a more complete manner, the same expression can be derived in the other extreme limit where the kinetic energy term is much higher than the spin-magnetic field interaction energy term.

IV. THE QUANTITATIVE ESTIMATES FOR SHOWING THE DEPARTURE

In the previous section we have shown that it is only when the kinetic energy associated with the wave function is much larger than the potential energy term in the Hamiltonian, in other words, the height of the potential barrier, or the depth of the potential well, we get back the standard expression for Larmor precession. However, when the kinetic energy term becomes comparable to the potential energy term due to the spin-magnetic field interaction, the standard expression no longer holds. In this section we calculate the effect of this deviation in

$v(m/sec)$	$p_+(\theta)$	$p'_+(\theta)$
2000	0.40725	0.40725
200	0.645427	0.645464
50	0.690242	0.653428
10	0.964184	0.855380

Table I: Table 1: This table shows the numerical values of $p_+(\theta)$ and $p'_+(\theta)$ for a fixed magnetic field, in this case, 2T, while decreasing the velocity of the incident neutrons. For thermal neutrons, we see that the two are the same, while differences start arising for cold neutrons and this difference is appreciable for ultra-cold neutrons.

$B(Tesla)$	$p_+(\theta)$	$p'_+(\theta)$
0.5	0.997736	0.912933
0.1	0.645427	0.660003
0.01	0.407245	0.407230
0.001	0.949661	0.949661

Table II: Table 2: This table shows the numerical values of $p_+(\theta)$ and $p'_+(\theta)$ for a fixed incident velocity of neutrons, in this case, in the ultra-cold neutron regime of $10m/s$, while decreasing the applied magnetic field.For a strong magnetic field, there is an appreciable difference between the two columns, which decreases as we decrease the magnetic field, till at low values of the field, the two are essentially the same.

terms of an observable, given by, in our case, the number of particles measured to be along $|\uparrow\rangle_\theta$ when the state emerging from the region of the magnetic field is passed through a Stern-Gerlach arrangement which is oriented at an angle θ with respect to the $+\hat{x}$ axis.

The state $|\uparrow\rangle_\theta$ is defined in the following manner $|\uparrow\rangle_\theta = 1/\sqrt{2}\ \left(|\uparrow\rangle_z + e^{i\theta}|\downarrow\rangle_z\right)$ The spin part of our original wavefunction is given by $\chi(0) = 1/\sqrt{2}\ \left(|\uparrow\rangle_z + |\downarrow\rangle_z\right)$. According to the standard treatment of Larmor precession, the final spin state is given by $\chi(\phi) = 1/\sqrt{2}\ \left(|\uparrow\rangle_z + e^{i\phi}|\downarrow\rangle_z\right)$. Therefore the probability of getting the particles with $|\uparrow\rangle_\theta$ is given by

$$p_+(\theta) = |_\theta\langle\uparrow|\ \chi(\phi)\rangle|^2 = \cos^2(\theta-\phi)/2 \tag{27}$$

However, in the light of the complete treatment presented in the last section, the final spin part of the wave function is given by Eq. (14) to be $\chi'(\phi) = 1/\sqrt{2}\ \left(ae^{i\phi_1}|\uparrow\rangle_z + be^{i\phi_2}|\downarrow\rangle_z\right)$. Consequently, the probability of getting particles with $|\uparrow\rangle_\theta$ is then modified which is of the form

$$p'_+(\theta) = \frac{1}{2}\left(a^2 + b^2 + 2ab\cos(\phi_1 - \phi_2 + \theta)\right) \tag{28}$$

It is clearly seen from Eqs.(27) and (28) that they are not same. We shall study the condition when they will be same. In the table given below, we compare the values of $p_+(\theta)$ and $p'_+(\theta)$ for $\theta = 0$ for different regimes of the velocity of the incident neutrons and the strength of the applied magnetic field. We can see clearly from the results given in the Tables 1 and 2 that when the incident velocity of the neutrons, or their kinetic energy is large, and the magnetic field is weak, $p_+(\theta)$ and $p'_+(\theta)$ are the same. However, on increasing the strength of the magnetic field, or decreasing the velocity of the incident neutrons, there is an empirically verifiable difference between $p_+(\theta)$ and $p'_+(\theta)$.

V. GENERALIZATION OF THE LARMOR PRECESSION TREATMENT FOR CALCULATING THE SPIN DISTRIBUTION FOR A WAVE PACKET

We will now use the results derived above to analyse the spin distribution which arises due to spin-magnetic field interaction when a wave packet passes through the spin rotator arrangement. For this purpose, we use a Gaussian wave packet as the space part of our initial wave function, instead of a plane wave as was used in Sections II and III. We choose the spin-polarization of the initial wave function to be in the $+x$ direction ,and the magnetic field to be pointing along the $+z$ direction. Thus our initial wave function is given by

$$|\Psi_i\rangle = \frac{1}{(2\pi\delta^2)^{\frac{1}{4}}} e^{-\frac{(x-x_0)^2}{4\delta^2}} e^{ik_0x} \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \tag{29}$$

where x_0 is the initial peak of the wavepacket, k_0 is the peak wave-number and δ is the width of the initial wavepacket. In the previous section we have seen that the precession of spin caused by interaction with the magnetic field within a spin rotator of given parameters is a function only of k. Therefore, while dealing with the Gaussian wave packet, it is convenient to use the Fourier transform of the initial wave function, given by

$$|\Psi_i\rangle = \left(\frac{2\delta^2}{\pi}\right)^{\frac{1}{4}} e^{-\delta^2(k-k_0)^2} e^{ikx_0} \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \tag{30}$$

Using results from the previous section, we can then write the final wave function in the Fourier basis to be

$$\begin{aligned}|\Psi_f\rangle \;=\; & \left(\frac{2\delta^2}{\pi}\right)^{\frac{1}{4}} e^{-\delta^2(k-k_0)^2} e^{ikx_0} \qquad (31)\\ & \times\frac{1}{\sqrt{2}}\left(a(k)e^{i\phi_1(k)}\,|\uparrow\rangle + b(k)e^{i\phi_2(k)}\,|\downarrow\rangle\right)\end{aligned}$$

where a, b, ϕ_1, ϕ_2 are as defined in Eqs. (15),(16),(17), and (18), and hence have different values for different values of k. From the above equation, it becomes clear that we have a spin distribution which occurs as a result of the spin-magnetic field interaction of different wave-number components of the original wave packet.

We will now find the distribution of spins along $|\chi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$ or the $+x$ direction. The projection of the spin of the final wave function along this direction is given by

$$\begin{aligned}\langle\chi\,|\Psi_f\rangle \;=\; & \left(\frac{2\delta^2}{\pi}\right)^{\frac{1}{4}} e^{-\delta^2(k-k_0)^2} e^{ikx_0} \qquad (32)\\ & \times\frac{1}{2}\left(a(k)e^{i\phi_1(k)} + b(k)e^{i\phi_2(k)}\right)\end{aligned}$$

Therefore, the probability of finding spins along the $+x$ direction will be given by

$$
\begin{aligned}
|\langle\chi\,|\Psi_f\rangle|^2 = \; & \left(\frac{2\delta^2}{\pi}\right)^{\frac{1}{2}} e^{-2\delta^2(k-k_0)^2} \\
& \times\frac{1}{4}\left(a(k)e^{i\phi_1(k)}+b(k)e^{i\phi_2(k)}\right) \\
& \times\left(a^*(k)e^{-i\phi_1(k)}+b^*(k)e^{-i\phi_2(k)}\right)
\end{aligned}
\tag{33}
$$

Such a spin probability function can be measured by a Stern Gerlach arrangement for the particles emerging from the spin rotator. A crucial feature to be stressed here is that the above spin probability function given by Eq. (33) has been obtained from our complete treatment of the problem of time evolution of the spin of a particle passing through a region of uniform magnetic field. On the other hand, the counterpart of such a spin probability distribution can also be obtained from the standard treatment of Larmor precession. A comparison between the results obtained from these two different approaches for various values of the magnetic field and velocity of the incident neutrons is illustrated in Figures 1 and 2. As similar to the case of the plane wave, we notice that the distributions from the two different approaches overlap when the incident velocity of the neutrons is high and the magnetic field is weak, whereas a noticeable divergence appears upon either decreasing the incident velocity or increasing the strength of the magnetic field.

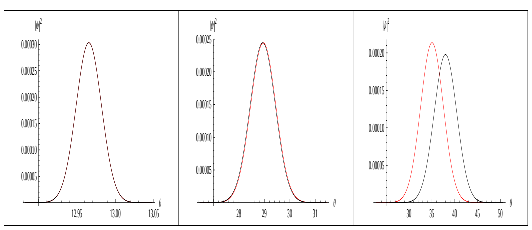

Figure 1: Here in the successive plots, we show the resultant probability distribution of spins emerging from the spin rotator, calculated according the standard Larmor precession formula(red curve) and the modified formula(black curve) given in this paper, while varying the constant magnetic field applied in the region of the spin rotator. The incident velocity is fixed at $10m/s$, while the applied magnetic field takes the values $0.001T$, $0.03T$ and $0.15T$. The observable effect of the departure from the standard expression becomes more pronounced as the magnetic field is increased.

VI. CONCLUSION AND OUTLOOK

In a nutshell, the central result of the present paper is that the standard expression for Larmor precession holds true only in the regime where the kinetic energy of the system is much greater as compared to the potential energy term arising out of the interaction between the spin of the particle and the magnetic field. It is essentially when these two terms become comparable that the de-

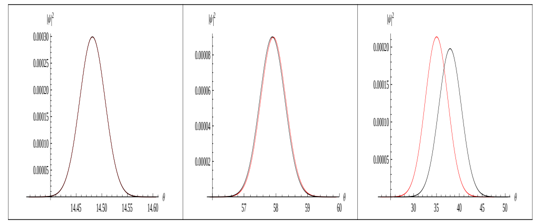

Figure 2: Here in the successive plots, we show the resultant probability distribution of spins emerging from the spin rotator, calculated according the standard Larmor precession formula(red curve) and the modified formula(black curve) given in this paper, while varying the incident velocity of the neutrons. The value of the applied magnetic field is held at $0.15T$, while the incident velocities are taken to be $100m/s$, $25m/s$ and $10m/s$. The observable effect of the departure from the standard expression becomes more pronounced when the incident velocity of the neutrons is decreased.

parture from the standard expression for Larmor precession becomes appreciable to the extent of being empirically observable - such a departure can indeed be tested by choosing appropriate conditions of high magnetic field and low velocity of incident neutrons; for example, in the experiments using cold neutrons.

An important application of the above treatment is that it enables the construction of an effective transit time distribution for a spin-polarised wave packet passing through a region of uniform magnetic field, of course, subject to the constraint of choosing the relevant parameters appropriately such that the rotation of spin pertaining to any wave-number component of the wavepacket does not exceed π. A rigorous derivation of the spin probability distribution emerging from a spin rotator is, therefore, a key ingredient for using Larmor precession in the context of its applicability as a model for quantum clock [13] that can be used for measuring arrival/transit time distributions, and for making a quantitative study of the possible differences in the predictions obtained from the different quantum mechanical schemes suggested for computing arrival/transit time distributions[14, 15]. Further studies along this line using the exact formula for Larmor precession derived in this paper are thus called for, the results of which could also be useful for comparing with other models for quantum clock suggested in the literature[16].

Among other possible uses of the exact formula for Larmor precession in the context of recent significant experiments, here we may mention, for example, the neutron ineterferometric experiment[17] testing single particle Bell's inequality [18]involving entanglement between the path and the spin degrees of freedom of a spin-1/2 particle. In such an experiment, the spin flipper that is placed in one of the two paths of the interferometer plays a crucial role for generating the path-spin entangled state, in the sense that the spin flipper is ideally required to flip all the neutrons passing through it, so that the flipped state and the unflipped spin state in the respective two paths are completely orthogonal. In order to ensure this condition, the choice of the relevant

parameters has to be carefully made on the basis of an appropriate formula for Larmor precession. Usually this is done by using the standard Larmor formula, as in the above mentioned experiment[17]. It is in this context that the exact formula for Larmor precession obtained in this paper should be of particular significance. For example, in the analysis of the above experiment, one needs to study the quantitative impact on the amount of violation of single particle Bell's inequality arising from the rigorous treatment of Larmor precession given in this paper.

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